(24) QN. → If $y = \log\left(\frac{x}{a+bx}\right)^x$, prove that

$$x^3 y_2 = (y - xy_1)^2$$

Ans. → ∵ $y = \log\left(\frac{x}{a+bx}\right)^x$

$\cdot\ y = x \log\left(\frac{x}{a+bx}\right) = x\left[\log x - \log(a+bx)\right]$

$\log\left(\frac{x}{a+bx}\right) = \frac{y}{x}$

$y = x\log x - x\log(a+bx)$

D.b.S.w.r.t. $x$, we have

$y_1 = \cancel{x}\times\frac{1}{\cancel{x}} + \log x \cdot 1 - \left[x \times \frac{1}{(a+bx)} \times b + \log(a+bx)\cdot 1\right]$

$= 1 + \log x - \left[\frac{bx}{(a+bx)} + \log(a+bx)\right]$

$= 1 + \log x - \frac{bx}{a+bx} - \log(a+bx)$

$= \left(1 - \frac{bx}{a+bx}\right) + \log x - \log(a+bx)$

$= \frac{a+\cancel{bx}-\cancel{bx}}{a+bx} + \log\frac{x}{a+bx}$

$y_1 = \frac{a}{a+bx} + \frac{y}{x}$ —— (1)

Again d.b.s.w.r.t. $x$, we have

$y_2 = \frac{a \times -1}{(a+bx)^2} \times b + \frac{xy_1 - y\times 1}{x^2}$

$y_2 = \frac{xy_1 - y}{x^2} - \frac{ab}{(a+bx)^2}$ —— (2)

multiplying by $x^3$, we have

$x^3 y_2 = x(xy_1 - y) - \frac{x^3 ab}{(a+bx)^2}$ —— (3)

From ①

$$y_1 - \frac{y}{x} = \frac{a}{a+bx}$$

$$\text{or, } \frac{xy_1 - y}{x} = \frac{a}{a+bx}$$

$$\text{or, } xy_1 - y = \frac{ax}{a+bx}$$

From ③

$$x^3 y_2 = x(xy_1 - y) - \frac{x^3 ab}{(a+bx)^2}$$

$$x^3 y_2 = \frac{x(ax)}{a+bx} - \frac{x^3 ab}{(a+bx)^2}$$

$$\text{or, } x^3 y_2 = \frac{ax^2}{a+bx} - \frac{x^3 ab}{(a+bx)^2}$$

$$= \frac{ax^2(a+bx) - x^3 ab}{(a+bx)^2}$$

$$= \frac{a^2x^2 + \cancel{abx^3} - \cancel{x^3 ab}}{(a+bx)^2}$$

$$= \frac{a^2x^2}{(a+bx)^2}$$

$$= \left(\frac{ax}{a+bx}\right)^2$$

$x^3 y_2 = (xy_1 - y)^2$ proved.

(25) Q.No.→ If $y = A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n$, prove that $(x^2-1)y_2 + xy_1 - n^2y = 0$.

Ans.→ $\because y = A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n$

D.b.S.w.r.t. $x$, we have

$$y_1 = nA(x+\sqrt{x^2-1})^{n-1} \times \left(1+\frac{1}{\cancel{2}\sqrt{x^2-1}}\times \cancel{2}x\right) +$$

$$Bn(x-\sqrt{x^2-1})^{n}\times\left(1-\frac{1}{\cancel{2}\sqrt{x^2-1}}\times \cancel{2}x\right)$$

$$y_1 = nA(x+\sqrt{x^2-1})^{n-1}\left(\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right) + Bn(x-\sqrt{x^2-1})^{n-1}$$

$$\times -\left(\frac{x-\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)$$

$$y_1 = nA\frac{(x+\sqrt{x^2-1})^n}{\sqrt{x^2-1}} - Bn\frac{(x-\sqrt{x^2-1})^n}{\sqrt{x^2-1}}$$

$$y_1\sqrt{x^2-1} = nA(x+\sqrt{x^2-1})^n - Bn(x-\sqrt{x^2-1})^n$$

Again D.b.S.w.r.t. $x$, we have

$$y_2\sqrt{x^2-1} + y_1\times\frac{1}{\cancel{2}\sqrt{x^2-1}}\times \cancel{2}x = n^2A\left[(x+\sqrt{x^2-1})^{n-1}\times\left(1+\frac{1}{\cancel{2}\sqrt{x^2-1}}\times\cancel{2}x\right)\right]$$

$$-Bn^2\left[(x-\sqrt{x^2-1})^{n-1}\times\left(1-\frac{1}{2\sqrt{x^2-1}}\times 2x\right)\right]$$

$$\text{or, } y_2\sqrt{x^2-1} + \frac{xy_1}{\sqrt{x^2-1}} = n^2A\left[\frac{(x+\sqrt{x^2-1})^n}{\sqrt{x^2-1}} + \frac{Bn^2(x-\sqrt{x^2-1})^n}{\sqrt{x^2-1}}\right]$$

$$y_2(x^2-1) + xy_1 = n^2\left[A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n\right]$$

$$y_2(x^2-1) + xy_1 = n^2[y]$$

$$y_2(x^2-1) + xy_1 = n^2y$$

or, $y_2(x^2-1)+xy_1-n^2y=0$ Proved

(27) QNo.→ If $y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$, Prove that

(i) $(x^2-1)y_2+xy_1-m^2y=0$,

and (ii) $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$

Ans.→ $\because y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$

D.b.S.w.r.t. $x$, we have

$$\frac{1}{m}y^{\frac{1}{m}-1}\cdot y_1-\frac{1}{m}y^{-\frac{1}{m}-1}\cdot y_1=2$$

or, $\frac{1}{m}\frac{y^{\frac{1}{m}}}{y}\cdot y_1-\frac{1}{m}\frac{y^{-\frac{1}{m}}}{y}\cdot y_1=2$

or, $\frac{1}{m}\frac{y_1}{y}\left[y^{\frac{1}{m}}-y^{-\frac{1}{m}}\right]=2$

or, $y_1\left(y^{\frac{1}{m}}-y^{-\frac{1}{m}}\right)=2my$

Squaring both sides, we have

$$y_1^2\left(y^{\frac{1}{m}} - y^{-\frac{1}{m}}\right)^2 = 4m^2y^2$$

$$y_1^2\left[\left(y^{\frac{1}{m}} + y^{-\frac{1}{m}}\right)^2 - 4\,y^{\frac{1}{m}}\cdot y^{-\frac{1}{m}}\right] = 4m^2y^2$$

Or, $y_1^2\left[4x^2 - 4\right] = 4m^2y^2$

$$\cancel{4}y_1^2(x^2-1) = \cancel{4}m^2 y^2$$

Again D.B.S. w.r.t. $x$, we have

$$2y_1y_2(x^2-1) + y_1^2 \times 2x = m^2\, 2y\,y_1$$

or, $y_2(x^2-1) + xy_1 = m^2 y$

Or, $y_2(x^2-1) + xy_1 - m^2y = 0$ [जब (i) रहे तब यहीं पर साबित हो गया
(ii) रहने पर शुरू से पूरा लिखना है

Diff. n times by Leibnitz's theorem, we have

$$y_{n+2}(x^2-1) + {}^nC_1\, y_{n+1}\, 2x + {}^nC_2\, y_n\, 2 + y_{n+1}\, x + {}^nC_1\, y_n \cdot 1 - m^2 y_n = 0$$

Or, $$y_{n+2}(x^2-1) + 2nx\,y_{n+1} + \frac{n(n-1)}{\cancel{2}}\cdot y_n \cancel{2} + x\,y_{n+1} + n y_n - m^2 y_n = 0$$

or, $(x^2-1)y_{n+2} + xy_{n+1}(2n+1) + y_n(n^2 - \cancel{n} + \cancel{n} - m^2) = 0$

or, $(x^2-1)y_{n+2} + xy_{n+1}(2n+1) + (n^2-m^2)y_n = 0$

Proved

(28) Q No → If $y = e^{a\sin^{-1}x}$, prove that

(i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Ans → ∵ $y = e^{a\sin^{-1}x}$

D.b.S.w.r.t. x, we have

$$y_1 = e^{a\sin^{-1}x} \times a \times \frac{1}{\sqrt{1-x^2}}$$

$$y_1 = \frac{ay}{\sqrt{1-x^2}}$$

or, $y_1\sqrt{1-x^2} = ay$

Squaring both sides, we have

$$y_1^2(1+x^2) = a^2y^2$$

Again diff. both s. w.r.t. x, we have

$$2y_1y_2(1-x^2) + y_1^2 \times -2x = a^2 \times 2yy_1$$

$$\cancel{2y_1}[y_2(1-x^2) - xy_1] = a^2\cancel{2}y\cancel{y_1}$$

$$y_2(1-x^2) - xy_1 = a^2y$$

or, $y_2(1-x^2) - xy_1 - a^2y = 0$ [जब (i) रहने पर स्थापित हो गया। (ii) रहने पर शुरूआती प्रश्न लिखना है।

$$(1-x^2)y_2 - xy_1 - a^2y = 0$$

Diff. n times by Leibnitz's theorem, we have

$$y_{n+2}(1-x^2) + {}^nc_1\, y_{n+1} \times -2x + {}^nc_2\, y_n \times -2 - [y_{n+1} \cdot x + {}^nc_1\, y_n]$$
$$- a^2 y_n = 0$$

$$\text{or, } y_{n+2}(1-x^2) - 2nx\, y_{n+1} - \frac{n(n-1)}{2} \times y_n \times 2 - y_{n+1} x - n y_n$$
$$n y_n - a^2 y_n = 0$$

$$\text{or, } y_{n+2}(1-x^2) - x y_{n+1}(2n+1) - [n^2 - \cancel{n} + \cancel{n} + a^2] y_n = 0$$

$$\text{or, } y_{n+2}(1-x^2) - x y_{n+1}(2n+1) - (n^2 + a^2) y_n = 0$$

Proved